IV. The Algorithm

1) Compute $A = p_\ell^2 - 1$; $B = p_m^2 - 1$.
2) If $|A| < \epsilon$ then $A = 0$.
3) If $|B| < \epsilon$ then $B = 0$.
4) If $x_\ell = x_m$ then $\mu_{-I} = -1$ and $\mu_I = (AB)^{1/2} + p_\ell p_m$.
5) If $x_\ell \neq x_m$ then
   a) replace a by $a+\epsilon$, and b by $b-\epsilon$.
   b) if $B \neq 0$ then $z^* = |A/B|^{1/2}$, else $z^* = 0$.
   c) If $JA < 0$ and $JB < 0$ and $z_a \leq z^* \leq z_b$ then $\mu_J' = -J\cdot(AB)^{1/2} + p_\ell p_m$ else $\mu_J' = J\cdot\max[J\cdot C(z_a), J\cdot C(z_b)]$.
   d) Set $\mu_J = J\cdot\min[1, J\cdot\mu_J']$; $\mu_{-J} = -J$.

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AIAA 81-4232

Growth and Decay of Weak Waves in Radiative Magnetogasdynamics

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Introduction

BECAUSE of the high ambient temperatures that prevail in many phenomena, it is of interest to consider the effects of thermal radiation in gasdynamics. Since at high temperatures a gas is likely to be at least partially ionized, electromagnetic effects can also be significant. One is thus led to study the interaction of the radiation and electromagnetic effects that may arise in problems of solar photosphere, rocket re-entry, and elsewhere. This Note extends the work of Singh and Sharma[1] to the magnetogasdynamic regime, and the effects of radiative transfer are treated by the use of a differential approximation which is valid over the entire optical depth range from the transparent limit (considered in Ref. 1) to the optically thick limit. The magnetic field is taken to be transverse to the direction of propagation of the wave.

Basic Equations

The fundamental equations are

$$(\partial\rho/\partial t) + u_i\rho_{,i} + \rho u_{i,i} = 0 \tag{1}$$

$$\rho(\partial u_i/\partial t) + \rho u_j u_{i,j} + (p+p_R)_{,i} + \mu H_j(H_{j,i} - H_{i,j}) = 0 \tag{2}$$

$$(\partial H_i/\partial t) + u_j H_{i,j} - H_j u_{i,j} + H_i u_{j,j} = 0 \tag{3}$$

$$\rho\left\{\partial\left(\frac{p}{\rho(\gamma-1)} + \frac{E_R}{\rho}\right)\Big/\partial t\right\} + \rho u_i\left(\frac{p}{\rho(\gamma-1)} + \frac{E_R}{\rho}\right)_{,i} + (p+p_R)u_{i,i} + q_{i,i} = 0 \tag{4}$$

$$p = \rho\mathcal{R}T \tag{5}$$

where q_i ($i=1,2,3$) and H_i are, respectively, the components of the radiative flux vector and the magnetic field strength, μ the magnetic permeability, and $\mathcal{R}$ the gas constant. The remaining symbols have the same definitions as in Ref. 1. The Maxwell equation $H_{i,i}$ is essentially included in Eq. (3) because the divergence of that equation gives $\partial(H_{i,i})/\partial t = 0$.

Within the differential approximation the equations of radiative transfer[2] may be written as the pair

$$(\partial E_R/\partial t) + q_{i,i} = -\bar{k}_p(cE_R - 4\sigma T^4) \tag{6}$$

$$(1/c)(\partial q_i/\partial t) + (c/3)E_{R,i} = -\bar{k}_p q_i \tag{7}$$

where $\bar{k}_p$ is the absorption coefficient depending on ρ and T, σ Stefan's constant, and c the velocity of light. Further, within the order of this approximation, the radiative pressure p_R is one-third of the radiative energy E_R.

Equations (1) and (4) yield

$$(\partial p/\partial t) + u_i p_{,i} + \rho a^2 u_{i,i} + 3(\gamma-1)\{(\partial p_R/\partial t) + u_i p_{R,i}\} + (\gamma-1)q_{i,i} = 0 \tag{8}$$

where $a = \{(\gamma p + 4(\gamma-1)p_R)/\rho\}^{1/2}$ is the effective acoustic speed.

A moving singular surface Σ, across which the flow parameters are essentially continuous but the discontinuities in their derivatives are permitted, is called a *weak wave*. Forming jumps across Σ in Eqs. (1-3) and (6-8) and using $u_n = 0$ and the compatibility conditions,[3] we get

$$G\zeta = \rho_0\lambda_i n_i \tag{9}$$

$$\rho_0 G\lambda_i = \xi n_i + \theta n_i + \mu H_j\eta_j n_i \tag{10}$$

$$G\eta_i = H_i\lambda_j n_j \tag{11}$$

$$3G\theta = \epsilon_i n_i \tag{12}$$

$$G\epsilon_i = c^2\theta n_i \tag{13}$$

$$G\xi - \rho_0 a_0^2\lambda_i n_i = (\gamma-1)(\epsilon_i n_i - 3G\theta) \tag{14}$$

where the subscript 0 denotes a value at the wave head; n_i is the unit normal vector; and $u_n = u_i n_i$, $\lambda_i = [u_{i,j}]n_j$, $\epsilon_i = [q_{i,j}]n_j$, $\eta_i = [H_{i,j}]n_j$, $\xi = [p_{,i}]n_i$, $\zeta = [\rho_{,i}]n_i$ and $\theta = [p_{R,i}]n_i$ are defined on Σ.

For an advancing wave surface, Eqs. (9-14), on using the fact that $G \neq 0$, yield

$$G = c/3^{1/2} \tag{15}$$

$$G = C_{\text{eff}_0} \tag{16}$$

Received July 14, 1980; revision received March 18, 1981.

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where $C_{\text{eff}_0} = (a_0^2 + b_0^2)^{1/2}$. The quantities a_0 and b_0 are, respectively, the acoustic speed and Alfven speed given by $a_0^2 = \{\gamma p_0 + 4(\gamma-1)p_{R_0}\}/\rho_0$ and $b_0^2 = \mu H_0^2/\rho_0$. Equations (15) and (16) represent the speeds of radiation-induced wave and the modified magnetogasdynamic wave, respectively.

Behavior of Radiation Induced Waves

When the first-order discontinuities propagate along the radiation-induced waves Σ, Eqs. (9-14), using Eq. (15), yield

$$\theta = \epsilon/3^{1/2}c \tag{17}$$

$$\zeta = 3^{1/2}\epsilon/c^3(1 - (3C_{\text{eff}_0}^2/c^2)) \tag{18}$$

$$\xi = 3^{1/2}a_0^2\epsilon/c^3(1 - (3C_{\text{eff}_0}^2/c^2)) \tag{19}$$

$$\lambda = \epsilon/\rho_0 c^2(1 - (3C_{\text{eff}_0}/c^2)) \tag{20}$$

where $\epsilon = \epsilon_i n_i$ and $\lambda = \lambda_i n_i$ are the wave amplitudes. Equation (5) together with Eqs. (18) and (19) gives

$$[T_{,i}]n_i = 3^{1/2}(a_0^2 - a_{T_0}^2)\epsilon/\mathfrak{R}\rho_0 c^3(1 - (3C_{\text{eff}_0}^2/c^2)) \equiv \chi \tag{21}$$

where $a_{T_0} = (p_0/\rho_0)^{1/2}$ is the isothermal speed of sound.

When Eq. (7) with the help of Eq. (6) is differentiated with respect to t and jumps are taken across Σ, we find, on using Eqs. (15) and (18-21) and the compatibility conditions,[3] that

$$(\delta\lambda/\delta t) + c(\bar{k}_p - 3^{-1/2}\Omega)\lambda = 0 \tag{22}$$

where the terms containing c^2 in the denominator are neglected. Equation (22) is the governing differential equation of the amplitude of a radiation-induced wave. The mean curvature Ω,[4] propagating with the constant velocity G, has the representation

$$\Omega = (\Omega_0 - K_0 G t)/(1 - 2\Omega_0 G t + K_0 G^2 t^2) \tag{23}$$

where $\Omega_0 = (k_1 + k_2)/2$ and $K_0 = k_1 k_2$ are, respectively, the mean and Gaussian curvatures of Σ at $t=0$ with k_1 and k_2 being the principal curvatures. When k_1 and k_2 are both nonpositive, the wave is divergent. If one or both of the principal curvatures are positive, the wave is convergent. Equation (22), using Eqs. (15) and (23), can be integrated to yield

$$\lambda = \lambda_0 I \exp(-\bar{k}_p c t) \tag{24}$$

where $I = \{(1 - 3^{-1/2}k_1 ct)(1 - 3^{-1/2}k_2 ct)\}^{-1/2}$, and λ_0 is the value of λ at $t=0$.

For diverging waves, Eq. (24) shows that $\lambda \to 0$ as $t \to \infty$, i.e., the radiation-induced waves are damped and the formation of a front carrying discontinuities in the flow quantities is not possible from a continuous flow. Equations (17-21) show that the quantities ζ, ξ, λ, and χ are small compared with ϵ and θ, as it should be in radiation-induced waves. For converging waves, there exists a finite time t^*, given by the smallest positive root of $(3^{1/2} - k_1 ct)(3^{1/2} - k_2 ct) = 0$, such that $\lambda \to \infty$ as $t \to t^*$. This corresponds to the formation of a focus.

Behavior of Modified Magnetogasdynamic Waves

Now consider discontinuities in the first derivatives of flow quantities propagating along a modified magnetogasdynamic wave Σ. The flow ahead of this wave will be disturbed by radiation-induced wave for which λ, ξ, ζ, and χ are small compared to θ and ϵ, and therefore the main effects on modified magnetogasdynamic waves will be derived for the special case in which $u_i = 0$ and p, ρ, and T are constant ahead of Σ. Equations (9-14), using Eq. (16), yield

$$\lambda = C_{\text{eff}_0}\xi/\rho_0 a_0^2 = C_{\text{eff}_0}\zeta/\rho_0 = C_{\text{eff}_0}\eta_i H_i/H_0^2 \tag{25}$$

$$\epsilon = \theta = 0 \tag{26}$$

When Eqs. (2), (3), and (6-8) are differentiated with respect to x_k and jumps are taken across Σ, we find, on using Eqs. (5), (16), (25), and (26) and the compatibility conditions,[3] that the resulting equations can be combined to yield

$$(\delta\lambda/\delta t) + (\Lambda_0 - C_{\text{eff}_0}\Omega)\lambda + \Gamma_0\lambda^2 = 0 \tag{27}$$

where

$$\Lambda_0 = \frac{(\gamma-1)}{2\mathfrak{R}\rho_0 C_{\text{eff}_0}^2}\Bigg[16\bar{k}_p\sigma T_0^3(a_0^2 - a_{T_0}^2)\{1 + C_{\text{eff}_0}(\gamma-1)^{-1}$$
$$\times (c^2 - C_{\text{eff}_0}^2)^{-1}\} - \left\{\mathfrak{R}\rho_0\left(\frac{\partial\bar{k}_p}{\partial\rho}\right)_0 + (a_0^2 - a_{T_0}^2)\left(\frac{\partial\bar{k}_p}{\partial T}\right)_0\right\}$$
$$\times\left\{(cE_{R_0} - 4\sigma T_0^4) + \frac{cq_n + C_{\text{eff}_0}(cE_{R_0} - 4\sigma T_0^4)}{(\gamma-1)(c^2 - 3C_{\text{eff}_0}^2)}\right\}\Bigg]$$

$$\Gamma_0 = \frac{(\gamma+1)a_0^2 + 3b_0^2}{2C_{\text{eff}_0}^2} > 0 \quad \text{and} \quad q_n = q_i n_i$$

Since $\bar{k}_p$ is an arbitrary function of ρ and T, the sign of Λ_0 may be positive or negative depending on the form of $\bar{k}_p$.

Equation (27) is the governing differential equation of the amplitude of a modified magnetogasdynamic wave. The solution of Eq. (27) is

$$\lambda = \lambda_0 \exp(-\Lambda_0 t) I_1(t)/(1 + \lambda_0\Gamma_0 I_2(t)) \tag{28}$$

where

$$I_1(t) = \{(1 - k_1 C_{\text{eff}_0}t)(1 - k_2 C_{\text{eff}_0}t)\}^{-1/2} \tag{29}$$

$$I_2(t) = \int_0^t I_1(\hat{t})\exp(-\Lambda_0\hat{t})\,d\hat{t} \tag{30}$$

The behavior of the wave amplitude depends upon the signs of k_1, k_2, and Λ_0. The following possibilities are easily verified from Eqs. (28) and (30).

Diverging Waves

Let $\Lambda_0 > 0$, from which I_2 converges at $t \to \infty$. If $\lambda_0 > 0$ (expansive wave), then $\lambda \to 0$ as $t \to \infty$ (i.e., the wave decays). If $\lambda_0 < 0$ (compressive wave), there is a critical amplitude λ_c given by

$$\lambda_c = \left\{\Gamma_0\int_0^\infty I_1(t)\exp(-\Lambda_0 t)\,dt\right\}^{-1}$$

such that if $|\lambda_0| < \lambda_c$, then $\lambda \to 0$ as $t \to \infty$; if $|\lambda_0| > \lambda_c$, the wave grows into a shock in a finite time t_c given by $I_2(t_c) = (\Gamma_0|\lambda_0|)^{-1}$; if $|\lambda_0| = \lambda_c$, then $\lambda \to (\Lambda_0/\Gamma_0)$ as $t \to \infty$, i.e., the wave takes a stable form. The derivatives of t_c with respect to Λ_0, $|k_1|$, and $|k_2|$ are positive; therefore, an increase in Λ_0, $|k_1|$, or $|k_2|$ delays the onset of the shock wave. Further, since Λ_0 decreases with increasing magnetic field, we infer that magnetic field effects hasten the onset of the shock wave.

Let $\Lambda_0 < 0$, from which $\lim_{t\to\infty} I_2(t) = \infty$. If $\lambda_0 > 0$, then $\lambda \to (|\Lambda_0|/\Gamma_0)$ as $t \to \infty$. If $\lambda_0 < 0$, all compressive waves, no matter how small their initial amplitude, always terminate into shock waves in a finite time $\bar{t}_c$ (i.e., $|\lambda| \to \infty$ as $t \to \bar{t}_c$) given by

$$\int_0^{\bar{t}_c} I_1(t)\exp(|\Lambda_0|t)\,dt = (\Gamma_0|\lambda_0|)^{-1}$$

The derivative of $\hat{t}_c$ with respect to $|\Lambda_0|$ is negative; therefore the thermal radiation effects with $\Lambda_0<0$ hasten the onset of a shock wave, whereas the magnetic field effects delay the onset of a shock wave.

Converging Waves

Let $k_1 \neq k_2$ and $\Lambda_0>0$, or $\Lambda_0<0$, from which $I_1 \to \infty$ as $t \to t^*$ and $I_2(t)$ tends to a finite value as $t \to t^*$ where t^* is a finite time given by the smallest positive root of $(1-k_1 C_{\mathrm{eff}_0} t)(1-k_2 C_{\mathrm{eff}_0} t)=0$. If $\lambda_0>0$, $|\lambda| \to \infty$ as $t \to t^*$, i.e., a focus is formed. If $\lambda_0<0$, there is a critical amplitude $\hat{\lambda}_c$ given by

$$\hat{\lambda}_c = \{\Gamma_0 I_2(t^*)\}^{-1}$$

such that if $|\lambda_0| < \hat{\lambda}_c$, then $|\lambda| \to \infty$ as $t \to t^*$, i.e., the focus forms but not the shock; if $|\lambda_0| > \hat{\lambda}_c$, then $|\lambda| \to \infty$ as $t \to \hat{t}_c$ where $\hat{t}_c$ is given by

$$I_2(\hat{t}_c) = (\Gamma_0 |\lambda_0|)^{-1}$$

Since $\hat{t}_c < t^*$, the shock wave is formed before the focus. If $|\lambda_0| = \hat{\lambda}_c$, the shock and focus form simultaneously.

Let $k_1 \neq k_2$. Then the singularity at t^* of the integrand in $I_2(t)$ is of the form $z^{-1}r(z)$ as $z \to 0$ where $r(z)$ is bounded away from zero. Hence $I_2(t) \to \infty$ as $t \to t^*$. If $\lambda_0>0$, then $|\lambda| \to \infty$ as $t \to t^*$, i.e., a focus is formed within a finite time t^*. If $\lambda_0<0$, the irrespective of the sign of Λ_0, there exists a finite time $\tilde{t}>0$ such that $\tilde{t}<t^*$, $I_2(\tilde{t}) = (\Gamma_0 |\lambda_0|)^{-1}$, and $\lim_{t \to \tilde{t}} |\lambda| = \infty$. Therefore, *any* converging compressive wave will grow into a shock wave before the formation of the focus, no matter how small its initial amplitude.

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